Recap: Computing f^e from W_m or W_m' relies on the system being conservative. Where have we used the fact that the system is conservative?

Recall that we used the line integral of dW_m or dW_m' to compute W_m or W_m', from which we compute f^e. Without the system being conservative, the line integral is not independent of the path taken, and our formulae for W_m or W_m' are not valid.

So far, we have computed W_m or W_m', given $\lambda(i, x)$, and then computed $f^e = -\frac{\partial W_m}{\partial x} = \frac{\partial W_m'}{\partial x}$.

Now, let's ask a different question. Given λ and W_m / W_m', we ask whether the system is conservative.

Recall that

$$dW_m = i(\lambda, x)\, d\lambda - f^e(\lambda, x)\, dx,$$

$$dW_m' = \lambda(i, x)\, di + f^e(i, x)\, dx.$$

Find a condition such that the path integral of dW_m or dW_m' is independent of path. Answer is given by Green's theorem.

<u>Green's Theorem:</u> Suppose $L : \mathbb{R}^2 \to \mathbb{R}$ such that

$$dL = F(x,y)\, dx + G(x,y)\, dy.$$

Then path integral of dL is independent of the path if and only if

$$\boxed{\frac{\partial F}{\partial y} = \frac{\partial G}{\partial x}}.$$

- Apply this to W_m & W_m'.

The system is conservative if and only if

I $\quad \dfrac{\partial i}{\partial x} = -\dfrac{\partial f^e}{\partial \lambda}$ or II $\quad \dfrac{\partial \lambda}{\partial x} = \dfrac{\partial f^e}{\partial i}$.

When applying this rule, make sure that f^e is expressed as a function of (λ, x) when applying I, and (i, x) when applying II.

<u>Example</u>: For a system, suppose we have

$$i = I_0 \left[\frac{\lambda/\lambda_0 + (\lambda/\lambda_0)^3}{1 + x/a} \right],$$

$$f^e = \frac{I_0}{a} \left[\frac{\frac{1}{2}(\lambda^2/\lambda_0) + \frac{1}{4}\lambda^4/\lambda_0^3}{(1 + x/a)^2} \right].$$

Is the system conservative? Here, λ_0, I_0 are constants.

$$\frac{\partial i^o}{\partial x} = \frac{\partial}{\partial x}\left\{ I_0 \left[\frac{\lambda/\lambda_0 + (\lambda/\lambda_0)^3}{1 + x/a} \right] \right\}$$

$$= -\frac{I_0}{a} \frac{\lambda/\lambda_0 + (\lambda/\lambda_0)^3}{(1 + x/a)^2}.$$

$$\frac{\partial f^e}{\partial \lambda} = -\frac{\partial}{\partial \lambda}\left\{ \frac{I_0}{a} \left[\frac{\frac{1}{2}(\lambda^2/\lambda_0) + \frac{1}{4}\lambda^4/\lambda_0^3}{(1 + x/a)^2} \right] \right\}$$

$$= -\frac{I_0}{a} \cdot \frac{1}{(1+x/a)^2} \left[\cancel{\tfrac{1}{2}} \cancel{2}\lambda/\lambda_0 + \cancel{\tfrac{1}{4}} \cancel{4}\lambda^3/\lambda_0^3 \right]$$

$$= -\frac{I_0}{a} \cdot \frac{\lambda/\lambda_0 + (\lambda/\lambda_0)^3}{(1 + x/a)^2}$$

$$= \frac{\partial i^o}{\partial x}.$$

Green's theorem $\Longrightarrow$ the system is conservative.

Green's Theorem: Suppose $L: \mathbb{R}^2 \to \mathbb{R}$ such that

$$dL = F(x,y)\,dx + G(x,y)\,dy.$$

Then path integral of dL is independent of the path if and only if

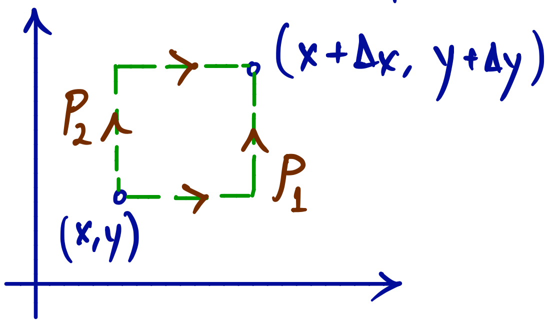
$$\boxed{\frac{\partial F}{\partial y} = \frac{\partial G}{\partial x}}.$$

You don't need to remember proofs.

Proof sketch of Green's theorem:

It's not a formal proof, but a sketch of why $\partial F/\partial y = \partial G/\partial x$ holds.

Consider two different paths from (x,y) to $(x+\Delta x, y+\Delta y)$ where $\Delta x, \Delta y$ are small.



Let's integrate dL along P_1 and P_2, and equate them.

$$\int_{P_1} dL = \int_{P_2} dL$$

Notice that $\displaystyle\int_{P_1} dL \approx F(x,y)\,\Delta x + G(x+\Delta x, y)\,\Delta y$.

Similarly, $\displaystyle\int_{P_2} dL \approx G(x,y)\,\Delta y + F(x, y+\Delta y)\,\Delta x$.

Equating them, we get

$$F(x,y)\,\Delta x + G(x+\Delta x, y)\,\Delta y = G(x,y)\,\Delta y + F(x, y+\Delta y)\,\Delta x$$

Rearranging the above equation, we get

$$\frac{G(x+\Delta x, y) - G(x,y)}{\Delta x} = \frac{F(x, y+\Delta y) - F(x,y)}{\Delta y}$$

Taking Δx & Δy to zero, we get

$$\frac{\partial G}{\partial x} = \frac{\partial F}{\partial y}.$$

Energy conversion cycles :

Recall that $dW_m = i\,d\lambda - f^e dx$. Suppose we go over a cycle $\mathcal{C}$ in the state space. Then, we have

$$\underbrace{\oint_{\mathcal{C}} dW_m}_{} = \underbrace{\oint_{\mathcal{C}} i\,d\lambda}_{} + \underbrace{\oint_{\mathcal{C}} -f^e dx}_{}.$$

Assuming the system is conservative, this term is zero.

Energy from mechanical port "into" the system, denoted by $EFM|_{\mathcal{C}}$.

Energy from electrical port "into" the system, denoted by $EFE|_{\mathcal{C}}$.

$$0 = \mathrm{EFE}\Big|_{\mathscr{C}} + \mathrm{EFM}\Big|_{\mathscr{C}} .$$

Two different cases may arise:

- $\mathrm{EFE}\Big|_{\mathscr{C}} > 0$ & $\mathrm{EFM}\Big|_{\mathscr{C}} < 0$

  This means the system absorbs electrical energy and performs mechanical work.

  ... The system performs motor action.

- $\mathrm{EFE}\Big|_{\mathscr{C}} < 0$ & $\mathrm{EFM}\Big|_{\mathscr{C}} > 0$

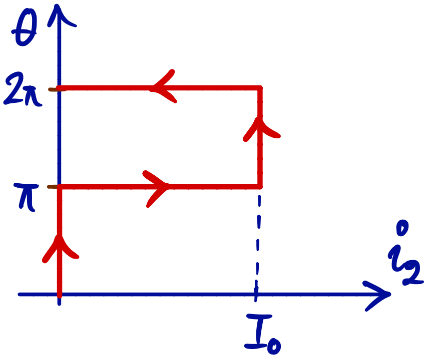
  This means the system absorbs mechanical energy and generates electrical energy.

  ... The system acts as a generator.

Example: Consider a rotational system, described as

$$\lambda_1(i_1, i_2, \theta) = L_{11}\, i_1 + M\cos\theta\, i_2,$$
$$\lambda_2(i_1, i_2, \theta) = M\cos\theta\, i_1 + L_{22}\, i_2.$$

Let's operate this system over the following cycle in (i_2, θ) space as shown, when i_1 is held constant at I_0, throughout. Assuming the system to be conservative, state whether the system is behaving as a generator or a motor.



Steps in the computation:

Alternately, calculate $EFE|_{cycle}$

1. Compute W_m or W_m' from λ_1, λ_2
2. Compute T^e from W_m or W_m'.
3. Compute $EFM|_{cycle} = \oint_{cycle} -T^e d\theta$.

Step 1:

$$W_m' = \int_0^{\mathring{i}_1} \lambda_1(\tilde{\mathring{i}}_1, 0, \theta)\, d\tilde{\mathring{i}}_1 + \int_0^{\mathring{i}_2} \lambda_2(\mathring{i}_1, \tilde{\mathring{i}}_2, \theta)\, d\tilde{\mathring{i}}_2$$

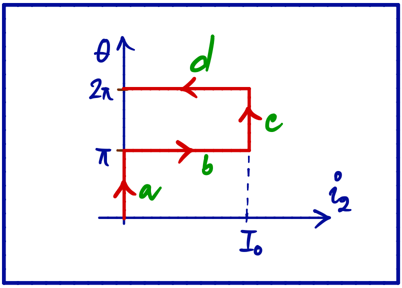
$$= \int_0^{\mathring{i}_1} L_{11}\, \tilde{\mathring{i}}_1\, d\tilde{\mathring{i}}_1 + \int_0^{\mathring{i}_2} (M\cos\theta\, \mathring{i}_1 + L_{22}\, \tilde{\mathring{i}}_2)\, d\tilde{\mathring{i}}_2$$

$$= L_{11}\frac{\mathring{i}_1^2}{2} + M\cos\theta\, \mathring{i}_1 \mathring{i}_2 + L_{22}\frac{\mathring{i}_2^2}{2}.$$

Step 2:

$$T^e = \frac{\partial W_m'}{\partial\theta} = -M\sin\theta\, \mathring{i}_1 \mathring{i}_2.$$

Step 3:



$$EFM\big|_{cycle} = \oint -T^e d\theta$$

$$= \int_a M\sin\theta\, \mathring{i}_1 \cancelto{=0}{\mathring{i}_2}\, d\theta$$

$$+ \int_b M\sin\theta\, \mathring{i}_1 \mathring{i}_2 \cancelto{=0}{d\theta} + \int_c M\sin\theta\, \underbrace{\mathring{i}_1}_{I_0}\, \underbrace{\mathring{i}_2}_{I_0}\, d\theta + \int_d M\sin\theta\, \mathring{i}_1 \mathring{i}_2 \cancelto{=0}{d\theta}$$

$$\therefore \ EFM\Big|_{cycle} = \int_{\theta=\pi}^{\theta=2\pi} M I_0^2 \sin\theta \, d\theta$$

$$= M I_0^2 \left(-\cos\theta\right)\Big|_{\theta=\pi}^{\theta=2\pi}$$

$$= -M I_0^2 \left(\cos 2\pi - \cos \pi\right).$$

$$= -M I_0^2 (1+1)$$

$$= -2 M I_0^2 .$$

$$< 0 .$$

$\therefore \ EFM\Big|_{cycle} < 0, \quad EFE\Big|_{cycle} > 0,$

i.e., it absorbs electrical energy & produces mechanical energy.

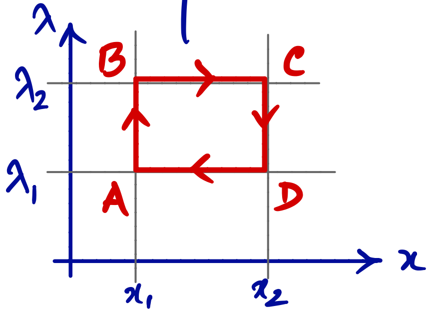
$\Rightarrow$ It is performing motor action.

- Which one among $EFE|_{cycle}$ and $EFM|_{cycle}$ will you calculate.

$$EFE\Big|_{cycle} = \oint_{cycle} \sum_{k} i_k \, d\lambda_k ,$$

where k ranges over all electrical ports.
Computing $\oint i\,d\lambda$ requires inversion of the λ-i relation, just the way you need when computing W_m. Usually, the route to compute $EFM\Big|_{cycle}$ is easier.

Example : Suppose $\lambda = L_0 \cdot \dfrac{i}{1+x/a}$ for a conservative system that is operated on the cycle shown in the figure. Is it behaving as a generator or a motor?



① Computing W_m' :

$$W_m' = \int_0^{\mathring{i}} \lambda(\tilde{\mathring{i}}, x)\, d\tilde{\mathring{i}} = \frac{L_0}{1+x/a} \int_0^{\mathring{i}} \tilde{\mathring{i}}\, d\tilde{\mathring{i}}$$

$$= \frac{L_0\, \mathring{i}^2}{2(1+x/a)}.$$

② Computing f^e :

$$f^e = \frac{\partial W_m'}{\partial x} = \frac{L_0\, \mathring{i}^2}{2} \cdot \frac{-1}{\left(1+\frac{x}{a}\right)^2} \cdot \frac{1}{a}.$$

$$= \frac{\cancel{L_0}}{2a} \cdot \frac{-1}{\cancel{\left(1+\frac{x}{a}\right)^2}} \cdot \frac{\lambda^2 \cancel{\left(1+\frac{x}{a}\right)^2}}{L_0^{\cancel{2}}}$$

$$= -\frac{\lambda^2}{2L_0 a}.$$

③ Computing $EFM\big|_{cycle}$:

$$EFM\big|_{cycle} = \int_{AB} -f^e \underbrace{\cancel{dx}}_{=0} + \int_{BC} -f^e dx + \int_{CD} -f^e \underbrace{\cancel{dx}}_{=0}$$

$$+ \int_{DA} -f^e dx$$

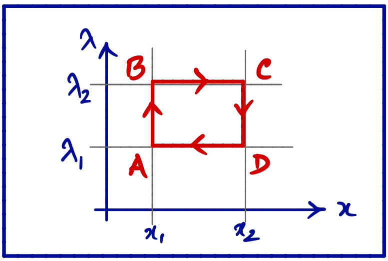
$$\therefore \text{EFM}\Big|_{cycle} = \int_{BC} -f^e dx + \int_{DA} -f^e dx$$

$$= \int_{x=x_1}^{x=x_2} \frac{+\lambda_2^2}{2L_0 a}\,dx + \int_{x=x_2}^{x=x_1} \frac{+\lambda_1^2}{2L_0 a}\,dx$$

$$= \frac{\lambda_2^2 (x_2 - x_1)}{2L_0 a} + \frac{\lambda_1^2 (x_1 - x_2)}{2L_0 a}.$$

$$= \frac{(x_2 - x_1)(\lambda_2^2 - \lambda_1^2)}{2L_0 a}$$

> 0 ... The system absorbs mechanical energy and produces electrical energy. It is a generator.



Exercise: Repeat it by calculating $\text{EFE}\big|_{cycle}$ for this problem.